Journal of Global Optimization (2005) 33: 369–391
DOI 10.1007/s10898-004-6098-5

On the Finite Termination of an Entropy Function Based Non-Interior Continuation Method for Vertical Linear Complementarity Problems

SHU-CHERNG FANG[1,2], JIYE HAN[3,*], ZHENG-HAI HUANG[4,**] and Ş. İLKER BİRBİL[5]
[1]*Industrial Engineering and Operations Research, North Carolina State University, Raleigh, NC 27695-7906, USA (e-mail: fang@ncsu.edu)*
[2]*Mathematical Sciences and Industrial Engineering, Tsinghua University, Beijing 100084, P.R. China*
[3]*Institute of Applied Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, P.R. China*
[4]*Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, P.R. China*
[5]*Faculty of Engineering and Natural Sciences, Sabancı University, Orhanli-Tuzla, 34956, Istanbul, Turkey*

(Received 8 August 2003; accepted in revised form 3 November 2004)

Abstract. By using a smooth entropy function to approximate the non-smooth max-type function, a vertical linear complementarity problem (VLCP) can be treated as a family of parameterized smooth equations. A Newton-type method with a testing procedure is proposed to solve such a system. We show that under some milder than usual assumptions the proposed algorithm finds an exact solution of VLCP in a finite number of iterations. Some computational results are included to illustrate the potential of this approach.

Mathematics Subject Classifications. 90C33, 65K10

Key words: Entropy function, Finite termination, Non-interior continuation method, Vertical linear complementarity problems, Smoothing approximation

1. Introduction

Let R^n be the space of n-dimensional real column vectors and $R^{m\times n}$ the space of $m\times n$ real matrices. We define an index set $\mathcal{I}:=\{1,2,\dots,n\}$. Given $N^i\in R^{m_i\times n}$, $q^i\in R^{m_i}$, and $s^i\in R^{m_i}$, with $m=\sum_{i=1}^n m_i\geqslant n$, define

*This author's work was partially supported by the National Natural Science Foundation of China (Grant Nos. 10271002 and 10401038).
**This author's work was partially supported by the Scientific Research Foundation of Tianjin University for the Returned Overseas Chinese Scholars and the Scientific Research Foundation of Liu Hui Center for Applied Mathematics, Nankai University-Tianjin University.

$$N := \begin{pmatrix} N^1 \\ N^2 \\ \vdots \\ N^n \end{pmatrix} \in R^{m\times n}, \quad q := \begin{pmatrix} q^1 \\ q^2 \\ \vdots \\ q^n \end{pmatrix} \in R^m, \quad \text{and } s := \begin{pmatrix} s^1 \\ s^2 \\ \vdots \\ s^n \end{pmatrix} \in R^m.$$

We say N is a vertical block matrix of type $(m_1, \dots, m_n)$. The vertical linear complementarity problem (VLCP) associated with N and q is to find a pair of vectors $x \in R^n$ and $s \in R^m$ such that

$$x \geqslant 0, \quad s^i = N^i x + q^i \geqslant 0, \text{ and } x_i \prod_{j=1}^{m_i} s_j^i = 0 \quad \forall i \in \mathcal{I}, \tag{1}$$

where x_i and s_j^i denote the ith component of x and the jth component of s^i, respectively. This problem was first introduced by Cottle and Dantzig in name of the *generalized linear complementarity problem* [5], since when $m_i = 1$ for all $i \in \mathcal{I}$, the problem reduces to an ordinary linear complementarity problem [6]. VLCP has various applications in non-linear networks [14], game theory [17], control theory [35] and economics [9]. Good references can be found in [8, 16, 27, 28, 30–32, 36].

Ebiefung [8] showed that VLCP is equivalent to a non-linear complementarity problem NCP(F) with $F = (F_1, \dots, F_n)^T$ and F_j, $j = 1, \dots, n$, being piecewise linear and concave. It can also be shown that VLCP is equivalent to a system of piecewise linear equations, or a multi-objective program. By extending Lemke's pivoting algorithm, Cottle and Dantzig proposed the first algorithm for VLCP [5]. An interior point method for solving extended vertical linear complementarity problems can be found in [38]. Peng and Lin [30] proposed a non-interior continuation method for solving VLCP. Qi and Liao [31] proposed a smoothing Newton method for extended VLCP.

In this paper, we are interested in developing a non-interior continuation method for solving VLCP with finite termination. Our approach is based on the entropic smoothing for the max-type function. Let $g_i\colon R^n \to R$, $\forall i \in \mathcal{I}$, be differentiable and define a max-type function $g\colon R^n \to R$ by

$$g(x) := \max_{i \in \mathcal{I}} g_i(x).$$

Although the function g is piecewise smooth and locally Lipschitz continuous, it is not differentiable. Given any $\mu > 0$, consider the following entropy-type function as a smoothing approximation function of g,

$$g(x, \mu) := \mu \ln \sum_{i=1}^{n} \exp(g_i(x)/\mu). \tag{2}$$

Note that, for $\mu > 0$,

$$g(x,\mu)=g(x)+\mu\ln\sum_{i=1}^{n}\exp\left(\frac{g_i(x)-g(x)}{\mu}\right). \tag{3}$$

Moreover,

$$g(x)\leqslant g(x,\mu)\leqslant g(x)+\mu\ln(n)\quad\forall x\in R^n \text{ and } \mu>0. \tag{4}$$

Therefore, $g(x,\mu)\to g(x)$ as $\mu\to 0$. This fact allows us to develop iterative methods based on $g(x,\mu)$ to solve the problem without facing the non-differentiability problem of $g(x)$. The function (2) was introduced by Kort and Bertsekas [24] as a penalty function for constrained minimization. Goldstein [15] studied this function intensively and attributed the basic approximation formula (4) to his former student Chang [3]. Since the function (2) can be derived from the dual problem of an entropy optimization problem [12], we call function (2) an entropic smoothing approximation function. Independently, Li [25] discovered a few properties of this function and named it as the *aggregate function*. Related work can be found in [1,3,11,15,26,30–32,41]. Also note that since a lower bound of the value of $g(x)$ is singled out in the representation (3), it can be used in computation to avoid the potential overflow problem arising from any exponential function evaluation in (2).

The finite termination of iterative methods is an interesting and important research topic. This property has been investigated for various cases including the interior point methods [23,29,42], non-smooth Newton methods [13,22,37], and non-interior continuation methods [4,7]. It is our objective to develop a Newton-type method based on the entropic smoothing function for solving VLCP in a finite number of iterations.

It is well-known that many non-interior continuation methods need to use the non-singularity assumption and the strict complementarity assumption to obtain the local superlinear convergence of the methods [2,19,20, 30,33]. The two assumptions mean that:

– Non-singularity assumption, i.e., the Jacobian matrix involved in Newton equation is non-singular at the solution point, or the iteration matrices are uniformly non-singular.
– Strict complementarity assumption, i.e., the solution of the problem concerned is strictly complementary.

The non-singularity assumption has recently been relaxed in a few non-interior continuation methods [10,21,40]. However, in order to achieve finite termination for non-interior continuation methods, this assumption has been commonly adopted, for example, see [4,7,13,37]. In this paper, by using the entropic approximation function, we present a non-interior continuation method for solving VLCP, in which a test procedure of finding a solution point in the optimal face of the problem is embedded into each iteration. We show that for N being a vertical block P_0 and R_0 matrix, if either (i) the strict complementarity condition holds, or (ii) the solution set of (1) is a singleton, then the proposed algorithm finds an exact solution to VLCP in a finite number of iterations. It should be noted that the commonly used non-singularity assumption implies that the solution set of the underlying problem is a singleton. Therefore, the hypothesis used in this paper is weaker.

The paper is organized as follows. Some basic concepts and properties for VLCP are introduced in Section 2. Then we present in Section 3 a non-interior continuation method based on the entropic approximation function for solving VLCP. In Section 4, we show the finite termination property of the proposed algorithm. Some numerical results are presented in Section 5.

2. Basic concepts and properties

A square matrix $M \in R^{n\times n}$ is said to be a P_0-matrix, if for all non-zero vector $x \in R^n$, there exists a component $x_i \neq 0$ such that $x_i(Mx)_i \geqslant 0$. For the vertical block matrix N of type $(m_1, \ldots, m_n)$, a square submatrix of N of order n is said to be a *representative submatrix*, if its ith row is drawn from the ith block N^i of N for each $i \in \mathcal{I}$. The following definition is from [28]:

DEFINITION 2.1. Let $N \in R^{m\times n}$ be a vertical block matrix of type $(m_1, \ldots, m_n)$. N is called a vertical block P_0-matrix, if all its representative submatrices are P_0-matrices. Moreover, N is called a vertical block R_0-matrix, if

$$\begin{pmatrix} \min\left\{x_1, N_1^1 x, \ldots, N_{m_1}^1 x\right\} \\ \vdots \\ \min\left\{x_n, N_n^1 x, \ldots, N_{m_n}^n x\right\} \end{pmatrix} = 0 \iff x = 0,$$

where N_j^i denotes the jth row of ith block.

For a VLCP with given N and q, it is obviously equivalent to the following piecewise smooth equations:

$$\begin{aligned} H(x) &:= \begin{pmatrix} \min\left\{x_1, N_1^1 x + q_1^1, \ldots, N_{m_1}^1 x + q_{m_1}^1\right\} \\ \vdots \\ \min\left\{x_n, N_1^n x + q_1^n, \ldots, N_{m_n}^n x + q_{m_n}^n\right\} \end{pmatrix} \\ &= -\begin{pmatrix} \max\left\{-x_1, -\left(N_1^1 x + q_1^1\right), \ldots, -\left(N_{m_1}^1 x + q_{m_1}^1\right)\right\} \\ \vdots \\ \max\left\{-x_n, -\left(N_1^n x + q_1^n\right), \ldots, -\left(N_{m_n}^n x + q_{m_n}^n\right)\right\} \end{pmatrix} = 0. \qquad (5) \end{aligned}$$

Let $\mathcal{F}$ be the feasible solution set of VLCP, i.e.,

$$\mathcal{F}=\{(x,s)\in R^n\times R^m: s=Nx+q\geqslant 0 \text{ and } x\geqslant 0\}.$$

Also let $\mathcal{S}$ denote the solution set of VLCP, i.e.,

$$\mathcal{S}=\{(x,s)\in R_n\times R_m: (x,s) \text{ satisfies (1)}\}.$$

Then we know that

$$(x,s)\in\mathcal{S} \quad \text{if and only if} \quad x \text{ solves } H(x)=0 \text{ and } s=Nx+q. \tag{6}$$

We also call $\mathcal{S}_1 := \{x \in R^n: H(x) = 0\}$ to be the solution set of VLCP. By applying the entropic approximation to $H(x)$, we can define a smooth function, for any $\mu > 0$,

$$H(x,\mu):=-\begin{pmatrix} \mu\ln\Big(\exp(-x_1/\mu)+\sum_{j=1}^{m_1}\exp\Big(-\Big(N_j^1x+q_j^1\Big)/\mu\Big)\Big) \\ \vdots \\ \mu\ln\Big(\exp(-x_n/\mu)+\sum_{j=1}^{m_n}\exp\Big(-\Big(N_j^nx+q_j^n\Big)/\mu\Big)\Big) \end{pmatrix}. \tag{7}$$

Consequently, $H(x,\mu) \to H(x)$ as $\mu \to 0$. This fact and (6) indicate that one can solve VLCP by taking the following steps: (i) start with a $\mu > 0$ and approximate VLCP by the parameterized smooth equations $H(x,\mu)=0$ and $s = Nx+q$, (ii) solve $H(x,\mu) = 0$ and maintain $s = Nx+q$ at each iteration, and (iii) refine the approximation by reducing the parameter μ to zero. Since it is usually very difficult to solve $H(x,\mu) = 0$ in an exact manner, for $\mu > 0$, like in other interior point and non-interior continuation methods, we use the following definition of neighborhood:

$$\mathcal{N}(\beta,\mu):=\{x\in R^n: \|H(x,\mu)\|\leqslant\beta\mu\}, \tag{8}$$

for $\beta > 0$ and $\mu > 0$.

The following lemma whose proof can be found in [30,31] summarizes some basic properties of the functions $H(x)$ and $H(x,\mu)$.

LEMMA 2.1. *Suppose that* $N \in R^{m\times n}$ *is a vertical block matrix of type* $(m_1,\ldots,m_n)$. *Let* $H(x)$ *and* $H(x,\mu)$ *be defined by* (5) *and* (7), *respectively. Then*

(i) *For each* $i\in\mathcal{I}$, $-H_i(x,\mu)$ *is convex and monotonically increasing with respect to* $\mu > 0$ *and*

$$-H_i(x)\leqslant -H_i(x,\mu)\leqslant -H_i(x)+\mu\ln(m_i+1),$$

where $H_i(x)$ *is the* i*th component of* $H(x)$ *as defined in (5).*

(ii) *If N is a vertical block P_0-matrix, then, for any $\mu > 0$, $-H_i(x, \mu)$ is an infinite order differentiable convex function with respect to $x \in R^n$, and $\nabla_x H(x, \mu)$ is non-singular for any $x \in R^n$ with*

$$\nabla_x H(x, \mu) = \begin{pmatrix} \lambda_0^1(x, \mu)e_1^T + \sum_{j=1}^{m_1} \lambda_j^1(x, \mu)N_j^1 \\ \lambda_0^2(x, \mu)e_2^T + \sum_{j=2}^{m_2} \lambda_j^2(x, \mu)N_j^2 \\ \cdots \\ \lambda_0^n(x, \mu)e_n^T + \sum_{j=1}^{m_n} \lambda_j^n(x, \mu)N_j^n \end{pmatrix},$$

where e_i is the ith column of the $n \times n$ identity matrix,

$$\begin{aligned}\lambda_0^i(x, \mu) &= \frac{\exp\left(-\frac{x_i}{\mu}\right)}{\exp\left(-\frac{x_i}{\mu}\right) + \sum_{l=1}^{m_i} \exp\left(-\frac{N_l^i x + q_l^i}{\mu}\right)} \\ &= \frac{\exp\left(\frac{-x_i + H_i(x)}{\mu}\right)}{\exp\left(\frac{-x_i + H_i(x)}{\mu}\right) + \sum_{l=1}^{m_i} \exp\left(\frac{-N_l^i x - q_l^i + H_i(x)}{\mu}\right)}, \quad i = 1, 2, \ldots, n, \\ \lambda_j^i(x, \mu) &= \frac{\exp\left(-\frac{N_j^i x + q_j^i}{\mu}\right)}{\exp\left(-\frac{x_i}{\mu}\right) + \sum_{l=1}^{m_i} \exp\left(-\frac{N_l^i x + q_l^i}{\mu}\right)} \\ &= \frac{\exp\left(-\frac{N_j^i x - q_j^i + H_i(x)}{\mu}\right)}{\exp\left(\frac{-x_i + H_i(x)}{\mu}\right) + \sum_{l=1}^{m_i} \exp\left(-\frac{N_l^i x - q_l^i + H_i(x)}{\mu}\right)}, \quad i = 1, 2, \ldots, n.\end{aligned}$$

(iii) *N is a vertical block R_0-matrix if and only if $\lim_{\|x\| \to \infty} \|H(x)\|/\|x\| \geqslant c_0$ holds for some constant $c_0 > 0$.*

(iv) *For any $x, y \in R^n$ and $\mu > 0$, there exists a constant $c_1 > 0$ such that*

$$\|H(y, \mu) - H(x, \mu) - \nabla_x H(x, \mu)(y - x)\| \leqslant \frac{\sqrt{n}c_1}{\mu}\|y - x\|^2.$$

(v) *For any $\mu_1, \mu_2 > 0$,*

$$\|H(x, \mu_1) - H(x, \mu_2)\| \leqslant \sqrt{n}(\ln \bar{m})|\mu_1 - \mu_2|,$$

where $\bar{m} = \max\{m_1, \ldots, m_n\} + 1$.

(vi) *If N is a vertical block R_0-matrix and $\mathcal{S}_1 \neq \emptyset$, then there exists a constant $c_2 > 0$ such that*

$$dist(x, \mathcal{S}_1) := \min_{y \in \mathcal{S}_1} \|y - x\| \leqslant c_2 \|H(x)\|,$$

for any $x \in R^n$.

Note that result (i) implies that $H(x,\mu) \to H(x)$ as $\mu \to 0$. Hence, we define $H(x,0) := H(x)$ for $x \in R^n$.

3. Proposed algorithm

Define an index set $\mathcal{J} := \{(i,j) : i \in \mathcal{I}, j = 1,2,\ldots,m_i\}$. The jth row N^i_j of N^i is called the (i,j) row of matrix N. Let $K_1 \subseteq \mathcal{I}$ and $K_2 \subseteq \mathcal{J}$ be two nonempty sets, then x_{K_1} and s_{K_2} denote the vectors obtained from all components x_r in x with $r \in K_1$ and all components s^i_j in s with $(i,j) \in K_2$, respectively. Moreover, $N_{K_2K_1}$ denotes the submatrix of N induced by those components of N whose row indices belong to K_2 and column indices belong to K_1, respectively. In what follows, k always denotes the iteration number.

ALGORITHM 3.1. Given $\sigma_1, \sigma_2 \in (0,1)$, $\alpha_1, \alpha_2 \in (0,1)$, $\gamma \in (0,1)$, $p \geqslant 1$, $\mu_0 > 0$, and $x^0 \in R^n$, choose $\beta > 0$ such that $\|H(x^0,\mu_0)\| \leqslant \beta\mu_0$. Set $s^0 := Nx^0 + q, k := 0$.

Step 1. If $H(x^k) = 0$, then stop (output x^k as a solution).
Step 2. If $\mu_k > \gamma$, then go to Step 3; otherwise, define four sets

$$\begin{aligned} A &:= \left\{ i \in \mathcal{I} : x^k_i > \sqrt{\mu_k} \right\}, & C &:= \left\{ i \in \mathcal{I} : x^k_i \leqslant \sqrt{\mu_k} \right\}, \\ B &:= \left\{ (i,j) \in \mathcal{J} : (s^k)^i_j > \sqrt{\mu k} \right\}, & D &:= \left\{ (i,j) \in \mathcal{J} : (s^k)^i_j \leqslant \sqrt{\mu k} \right\}. \end{aligned} \tag{9}$$

If one of the following four cases occurs, then stop (output x^{k+1} as a solution); otherwise, go to Step 3.

Case (i) If $A \neq \emptyset, B \neq \emptyset$ and for any $i \in A$ there exists at least one index $(i,j) \in \mathcal{J}$ such that $(s^k)^i_j \leqslant \sqrt{\mu_k}$, then solve the following system of equations:

$$\begin{pmatrix} s^k_B + \Delta S^k_B \\ 0 \end{pmatrix} = \begin{pmatrix} N_{BA} & N_{BC} \\ N_{DA} & N_{DC} \end{pmatrix} \begin{pmatrix} x^K_A + \Delta x^K_A \\ 0 \end{pmatrix} + \begin{pmatrix} q_B \\ q_D \end{pmatrix}. \tag{10}$$

If there exists a solution $(\Delta x^k_A, \Delta s^k_B)$ such that $x^k_A + \Delta x^k_A \geqslant 0$ and $s^k_B + \Delta s^k_B \geqslant 0$, then set

$$x^{k+1}_i := \begin{cases} x^k_i + \Delta x^k_i & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases},$$

$$(s^{k+1})^i_j := \begin{cases} (s^k)^i_j + (\Delta s^k)^i_j & \text{if } (i,j) \in B \\ 0 & \text{otherwise} \end{cases}.$$

Case (ii) If $A \neq \emptyset$ and $B = \emptyset$, then solve the following system of equations:

$$0 = (N_{DA}, N_{DC}) \begin{pmatrix} x_A^k + \Delta x_A^k \\ 0 \end{pmatrix} + q_D.$$

If there exists a solution Δx_A^k such that $x_A^k + \Delta x_A^k \geqslant 0$, then set

$$x_i^{k+1} := \begin{cases} x_i^k + \Delta x_i^k & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}, \quad s^{k+1} := 0.$$

Case (iii) If $A = \emptyset$, $B \neq \emptyset$ and $q_B > 0$, $q_D = 0$, then set

$$x^{k+1} := 0, \quad (s^{k+1})_j^i := \begin{cases} q_j^i & \text{if } (i, j) \in B \\ 0 & \text{otherwise} \end{cases}.$$

Case (iv) If $A = \emptyset$, $B = \emptyset$ and $q_D = 0$, then set $x^{k+1} := 0, s^{k+1} := 0$.

Step 3. Find a Newton direction Δx^k by solving $\nabla_x H(x^k, \mu_k)\Delta x^k = -H(x^k, \mu_k)$. Let θ_k be the maximum value of the set $\{1, \alpha_1, \alpha_1^2, \dots\}$ such that

$$\|H(x^k + \theta_k \Delta x^k, \mu_k)\| \leqslant (1 - \sigma_1 \theta_k)\beta\mu_k.$$

Set $x^{k+1} := x^k + \theta_k \Delta x^k$. Moreover, let λ_k be the maximum value of the set $\left\{\max\left\{1, \frac{1}{\sigma_2}\left(1 - \mu_k^p\right)\right\}, \alpha_2, \alpha_2^2, \dots\right\}$ such that

$$x^k + \theta_k \Delta x^k \in \mathcal{N}(\beta, (1 - \sigma_2 \lambda_k)\mu_k).$$

Set $\mu_{k+1} := (1 - \sigma_2 \lambda_k)\mu_k$.

Step 4. Set $s^{k+1} := Nx^{k+1} + q$ and $k := k + 1$. Go to Step 1.

Note that since the Jacobian matrix $\nabla_x H(x, \mu)$ in Step 3 is guaranteed to be non-singular for any $\mu > 0$ and $x \in R^n$ by the result (ii) of Lemma 2.1, it is not difficult to see that Algorithm 3.1 is well-defined. The initial value of parameter p ($\geqslant 1$) can be selected to be a suitable positive integer. The parameter γ in Step 2 is used to control the quality of final solution. In the next section (see Lemma 4.4), we show that the four index sets (9) in Step 2 actually coincide with the index sets of a solution to VLCP as k becomes sufficiently large.

THEOREM 3.1. *If Algorithm 3.1 terminates in either Step 1 or Step 2 for some $k \geqslant 0$, then (x^k, s^k) or (x^{k+1}, s^{k+1}) is a solution to VLCP, respectively.*

Proof. If Algorithm 3.1 terminates in Step 1, that is,

$$H(x^k) = 0 \tag{11}$$

for some $k \geqslant 0$. From the algorithm, it is easy to see that

$$s^k = Nx^k + q \tag{12}$$

for all $k \geqslant 0$. From (6), (11) and (12), it follows that (x^k, s^k) is a solution to VLCP.

If one stopping criterion in Step 2 is met: since A and C form a partition of $\mathcal{I}$, and B and D form a partition of $\mathcal{J}$, it is not difficult to check from cases (i)–(iv) that x^{k+1} and s^{k+1} satisfy the non-negativity condition, feasibility condition, and complementarity condition of the system (1). Consequently, (x^{k+1}, s^{k+1}) is a solution to VLCP. □

A non-interior continuation method in general generates a sequence of infinitely many iterations. In this case, only an approximate solution is generated. But if the proposed algorithm terminates in either Step 1 or Step 2 for some $k \geqslant 0$, then Theorem 3.1 guarantees an exact solution to VLCP.

4. Finite termination

In this section we show that, under some milder than usual conditions, the stopping criteria in Step 2 of the proposed algorithm must be met as k becomes sufficiently large. This implies that the proposed algorithm terminates in a finite number of iterations.

THEOREM 4.2. *Let N be a vertical block P_0 and R_0 matrix of type $(m_1, \ldots, m_n)$ and $\{(x^k, s^k, \mu_k)\}$ be the sequence generated by Algorithm 3.1. If $H(x^k) \neq 0$ for $k = 0, 1, 2, \ldots,$ then*

- (i) *$\{(x^k, s^k, \mu_k)\}$ is a bounded infinite sequence,*
- (ii) *each accumulation point of the sequence $\{(x^k, s^k)\}$ is a solution to VLCP.*

Proof.

(i) If $H(x^k) \neq 0$ for all $k \geqslant 0$, then the proposed algorithm will not terminate at Step 2. Otherwise, if the algorithm terminates at Step 2 in $k_0 \geqslant 0$ iterations, then from Theorem 3.1, we know that $(x^{k_0+1}, s^{k_0+1}) \in S$, and hence $H(x^{k_0+1}) = 0$. Therefore an infinite sequence $\{(x^k, s^k, \mu_k)\}$ is generated. From the algorithm itself, it is not difficult to see that $s^k = Nx^k + q$ and $x^k \in \mathcal{N}(\beta, \mu_k)$ for $k = 0, 1, 2, \ldots$. Now, for any $x^k \in \mathcal{N}(\beta, \mu_k)$, the result (i) of Lemma 2.1 implies that

$$\|H(x^k)\| \leqslant \|H(x^k) - H(x^k, \mu_k)\| + \|H(x^k, \mu_k)\| \leqslant (\sqrt{n} \ln \bar{m} + \beta)\mu_0,$$

where $\bar{m} = \max\{m_1, \ldots, m_n\} + 1$. This inequality and the result (iii) of Lemma 2.1 further imply that $\{x^k\}$ is bounded. Consequently, $\{s^k\}$ is bounded because $s^k = Nx^k + q$. Note that $\{\mu_k\}$ obtained in Step 3 is a monotonically decreasing non-negative sequence. Hence the sequence $\{(x^k, s^k, \mu_k)\}$ is bounded.

(ii) Since the infinite sequence $\{(x^k, s^k, \mu_k)\}$ is bounded, there exists a convergent subsequence. We may assume without loss of generality that $\lim_{k\to\infty}(x^k, s^k, \mu_k) = (x^*, s^*, \mu_*)$. Because $s^k = Nx^k + q$ holds for all $k \geqslant 0$, we have

$$s^* = Nx^* + q. \tag{13}$$

Noting that $\{\mu_k\}$ is a monotonically decreasing non-negative sequence, we know $\mu_* \geqslant 0$. If $\mu_* = 0$, the result (i) of Lemma 2.1 implies that $H(x^*) = H(x^*, \mu_*)$. Moreover, $x^k \in \mathcal{N}(\beta, \mu_k)$ implies that $x^* \in \mathcal{N}(\beta, \mu_*)$. Hence $\|H(x^*, \mu_*)\| = 0$ and $H(x^*) = 0$. Together with (13), we know that $(x^*, s^*) \in \mathcal{S}$, and the desired result follows. We now show that $\mu_* > 0$ will not occur. Assume that $\mu_* > 0$, then the result (ii) of Lemma 2.1 implies that $\nabla_x H(x^k, \mu_k)$ is non-singular and its norm is uniformly bounded below by a positive constant for all $k \geqslant 0$. In other words, there exists a constant $c_3 > 0$ such that $\|\nabla_x H(x^k, \mu_k)]^{-1}\| \leqslant c_3$. By Step 3 of the proposed algorithm, we have

$$\begin{aligned}\|\Delta x^k\| &= \|[\nabla_x H(x^k, \mu_k)]^{-1} H(x^k, \mu_k)\| \\ &\leqslant c_3 \|H(x^k, \mu_k)\| \leqslant c_3 \beta \mu_k \ \text{ for } k \geqslant 0.\end{aligned} \tag{14}$$

For $\alpha \in (0, 1)$, define

$$r^k(\alpha) := H(x^k + \alpha \Delta x^k, \mu_k) - H(x^k, \mu_k) - \alpha \nabla_x H(x^k, \mu_k) \Delta x^k. \tag{15}$$

It follows from the result (iv) of Lemma 2.1 and (14) that

$$\|r^k(\alpha)\| \leqslant \frac{\sqrt{n}\alpha^2 c_1}{\mu_k} \|\Delta x^k\|^2 \leqslant \sqrt{n}\alpha^2 c_1 c_3^2 \beta \|H(x^k, \mu_k)\|.$$

If we let $\bar{\alpha} = \min\left\{\frac{1-\sigma_1}{\sqrt{n}\beta c_1 c_3^2}, 1\right\}$, then

$$\|r^k(\alpha)\| \leqslant (1 - \sigma_1)\alpha \|H(x^k, \mu_k)\| \quad \text{for any } \alpha \in (0, \bar{\alpha}). \tag{16}$$

Combining (15) and (16), we see

$$\begin{aligned}
&\|H(x^k+\alpha\Delta x^k,\mu_k)\|-(1-\sigma_1\alpha)\|H(x^k,\mu_k)\|\\
&\quad\leqslant(1-\alpha)\|H(x^k,\mu_k)\|+\|r^k(\alpha)\|-(1-\sigma_1\alpha)\|H(x^k,\mu_k)\|\\
&\quad=(\sigma_1-1)\alpha\|H(x^k,\mu_k)\|+\|r^k(\alpha)\|\\
&\quad\leqslant 0.
\end{aligned}\tag{17}$$

Let l_1 be the minimum integer such that $\alpha_1^{l_1}\leqslant\bar{\alpha}$, then $\theta_k\geqslant\alpha_1^{l_1}$ follows from the algorithm. Hence there exists a constant $\theta_*>0$ such that $\theta_k\geqslant\theta_*$ for all $k\geqslant 0$. It follows from (17) that

$$\|H(x^{k+1},\mu_k)\|\leqslant(1-\sigma_1\theta_k)\|H(x^k,\mu_k)\|\leqslant(1-\sigma_1\theta_*)\|H(x^k,\mu_k)\|.$$

Using the result (v) of Lemma 2.1 and the above inequality, we know that, for any $\lambda\in(0,1)$,

$$\begin{aligned}
\frac{\|H(x^{k+1},(1-\sigma_2\lambda)\mu_k)\|}{(1-\sigma_2\lambda)\mu_k}&\leqslant\frac{\|H(x^{k+1},\mu_k)\|+\sqrt{n}(\ln\bar{m})\lambda\sigma_2\mu_k}{(1-\sigma_2\lambda)\mu_k}\\
&\leqslant\frac{(1-\sigma_1\theta_*)\|H(x^k,\mu_k)\|+\sqrt{n}(\ln\bar{m})\lambda\sigma_2\mu_k}{(1-\sigma_2\lambda)\mu_k}\\
&\leqslant\frac{(1-\sigma_1\theta_*)\beta+\sqrt{n}(\ln\bar{m})\lambda\sigma_2}{1-\sigma_2\lambda}.
\end{aligned}$$

For $\frac{(1-\sigma_1\theta_*)\beta+\sqrt{n}(\ln\bar{m})\lambda\sigma_2}{1-\sigma_2\lambda}\leqslant\beta,\lambda\leqslant\bar{\lambda}:=\frac{\sigma_1\theta_*\beta}{\sqrt{n}(\ln\bar{m})\sigma_2+\sigma_2\beta}$. If we let l_2 be the minimum integer such that $\alpha_2^{l_2}\leqslant\min\{\bar{\lambda},1\}$, then a similar argument assures that there exists a constant $\lambda_*>0$ such that $\lambda_k\geqslant\lambda_*$ for any $k\geqslant 0$. In this case, $\mu_{k+1}=(1-\sigma_2\lambda_k)\mu_k\leqslant(1-\sigma_2\lambda_*)\mu_k$, which further implies that $\mu_k\to 0$ as $k\to\infty$. This contradicts the hypothesis of $\mu_*>0$. □

LEMMA 4.2. *Let N be a vertical block P_0 and R_0 matrix of type $(m_1,\dots,m_n)$. Then the solution set $\mathcal{S}$ of VLCP is non-empty and compact.*

Proof. Since N be a vertical block P_0 and R_0 matrix of type $(m_1,\dots,m_n)$, Theorems 3.1 and 4.2 imply that $\mathcal{S}$ is non empty. In addition, it is not difficult to see that $\mathcal{S}$ is closed. Thus, it suffices to show that $\mathcal{S}$ is bounded. If not, then there exists an unbounded solution sequence $\{(x^r,s^r)\}\in\mathcal{S}$ for all $r\geqslant 0$. It follows from (6) that

$$H(x^r)=0\quad\text{and}\quad s^r=Nx^r+q,\quad\text{for all } r\geqslant 0.$$

Consequently, $\lim_{r\to\infty}\|H(x^r)\|/\|x^r\|=0$. However, the result (iii) of Lemma 2.1 shows that there exists a constant $c_0>0$ such that $\lim_{r\to\infty}\|H(x^r)\|/\|x^r\|\geqslant c_0$. This contradicts the hypothesis. □

When N is a vertical block P_0 and R_0 matrix of type $(m_1, \dots, m_n)$, if $H(x^k) \neq 0$ for all $k \geqslant 0$, the result (i) of Theorem 4.2 says that Algorithm 3.1 generates a bounded infinite sequence $\{(x^k, s^k, \mu_k)\}$. Let $\{(x^{\bar{k}}, s^{\bar{k}}, \mu_{\bar{k}})\}$ be a convergent subsequence with a limit point (x^*, s^*, μ_*). Then the result (ii) of Theorem 4.2 says that $\mu_* = 0$ and $(x^*, s^*) \in \mathcal{S}$. By making use of the sequence $\{(x^{\bar{k}}, s^{\bar{k}}, \mu_{\bar{k}})\}$, for each $\bar{k} \geqslant 0$, we define four index sets:

$$A_{\bar{k}} := \left\{ i \in \mathcal{I} : x_i^{\bar{k}} > \sqrt{\mu_{\bar{k}}} \right\}, \quad C_{\bar{k}} := \left\{ i \in \mathcal{I} : x_i^{\bar{k}} \leqslant \sqrt{\mu_{\bar{k}}} \right\},$$

$$B_{\bar{k}} := \left\{ (i, j) \in \mathcal{J} : (s^{\bar{k}})_j^i > \sqrt{\mu_{\bar{k}}} \right\}, \quad D_{\bar{k}} := \left\{ (i, j) \in \mathcal{J} : (s^{\bar{k}})_j^i \leqslant \sqrt{\mu_{\bar{k}}} \right\}. \tag{18}$$

Clearly, $A_{\bar{k}}$ and $C_{\bar{k}}$ partition the index set $\mathcal{I}$ and $B_{\bar{k}}$ and $D_{\bar{k}}$ partition the index set $\mathcal{J}$. In what follows, we discuss the finite termination of Algorithm 3.1 in two cases.

Case 1: Finite Termination under Strict Complementarity

Let (x^*, s^*) be a solution to VLCP, for any $i \in \mathcal{I}$, we denote x_i^* by $(s^*)_0^i$, then $H_i(x^*) = \min\left\{(s^*)_0^i, (s^*)_1^i, \dots, (s^*)_{m_i}^i\right\}$. Let $I_i(x^*)$ denote the active set at x^* defined by $I_i(x^*) := \{j : H_i(x^*) = (s^*)_j^i, j = 0, 1, \dots, m_i\}$. Assume that (x^*, s^*) satisfies the strict complementarity condition, i.e., the cardinality of $I_i(x^*)$ is equal to one for all $i \in \mathcal{I}$. Define

$$\begin{aligned} \mathcal{B} &:= \left\{ i \in \mathcal{I} : x_i^* = 0 \right\}, \\ \mathcal{N} &:= \Big\{ i \in \mathcal{I} : x_i^* > 0, (s^*)_{j_{i_0}}^i = 0 \quad \text{for some } j_{i_0} \text{ and} \\ &\qquad (s^*)_j^i > 0 \quad \text{for all } j \neq j_{i_0}, 1 \leqslant j \leqslant m_i \Big\}. \end{aligned}$$

Since $(x^*, s^*) \in \mathcal{S}$ satisfies the strict complementarity condition, it follows that $\mathcal{B} \cup \mathcal{N} = \mathcal{I}$ and $\mathcal{B} \cap \mathcal{N} = \emptyset$.

For any $w = (x, s) \in R^n \times R^m$, let $\mathcal{N}_0 := \left\{ j : (s^*)_j^i = 0 \text{ with } i \in \mathcal{N} \right\}$ and let $s_{\mathcal{N}_0}^{\mathcal{N}}$ denote a vector with ith component being $s_{j_{i_0}}^i$ for $i \in \mathcal{N}$ and $j_{i_0} \in \mathcal{N}_0$. Define

$$G(w) := \begin{pmatrix} s - Nx - q \\ x_{\mathcal{B}} \\ s_{\mathcal{N}_0}^{\mathcal{N}} \end{pmatrix} \tag{19}$$

and

$$\mathcal{S}_0 := \{ w \in R^{n+m} : G(w) = 0 \}. \tag{20}$$

Similar to Lemma 5.1 in [21], we have the following result:

LEMMA 4.3. *Let*

$$\varepsilon = \min\left\{\min_{i\in\mathcal{N}}\left\{x_i^*, \min_{1\leqslant j\leqslant m_i, j\neq j_{i_0}}\left\{(s^*)_j^i\right\}\right\}, \min_{i\in\mathcal{B}, 1\leqslant j\leqslant m_i}\left\{(s^*)_j^i\right\}\right\}$$

and

$$\Delta = \big\{w = (x, s) \in R^n \times R^m : |x_i - x_i^*| \leqslant \varepsilon/3, |s_j^i - (s^*)_j^i| \leqslant \varepsilon/3, i \in \mathcal{I}, (i, j) \in \mathcal{J}\big\}.$$

Then for any $w \in \Delta \cap \mathcal{F}$ *there exists a constant* $\lambda > 0$ *such that*

$$\|H(x)\| = \|G(w)\| \geqslant \lambda \cdot \operatorname{dist}(w, \mathcal{S}_0), \tag{21}$$

where $H(\cdot)$ *and* $G(\cdot)$ *are defined by* (5) *and* (19), *respectively.*

Proof. Denote (x^*, s^*) by w^*. Since $G(w^*) = 0, G(w) = 0$ is solvable and $\mathcal{S}_0 \neq \emptyset$. By Hoffman's result on error bound of linear systems [18], there exists a positive number $\lambda > 0$ such that for any $w \in R^{n+m}$

$$\|G(w)\| \geqslant \lambda \cdot \operatorname{dist}(w, \mathcal{S}_0). \tag{22}$$

For any $w \in \Delta$, if $i \in \mathcal{N}$, then

$$\begin{aligned}
&x_i = x_i^* + x_i - x_i^* \geqslant \varepsilon - \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon,\\
&|s_{j_{i_0}}^i| \leqslant \frac{1}{3}\varepsilon,\\
&s_j^i = (s^*)_j^i + s_j^i - (s^*)_j^i \geqslant \varepsilon - \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon \quad \text{for } 1 \leqslant j \leqslant m_i, \quad j \neq j_{i_0}.
\end{aligned}$$

If $i \in \mathcal{B}$, then

$$\begin{aligned}
&|x_i| \leqslant \frac{1}{3}\varepsilon,\\
&s_j^i = (s^*)_j^i + s_j^i - (s^*)_j^i \geqslant \varepsilon - \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon \quad \text{for } 1 \leqslant j \leqslant m_i.
\end{aligned}$$

These imply that for each $i \in \mathcal{I}$,

$$\min\{x_i, s_1^i, s_2^i, \ldots, s_{m_i}^i\} = \begin{cases} x_i, & \forall i \in \mathcal{B} \\ s_{j_{i0}}^i, & \forall i \in \mathcal{N}. \end{cases}$$

Hence,

$$\|H(x)\| = \|G(w)\| \text{ for } w \in \Delta \cap \mathcal{F}. \tag{23}$$

(21) follows from (23) and (22). □

Lemma 4.3 indicates that if $w \in \mathcal{S}_0$ for w being sufficiently close to w^*, then w solves (1), i.e., $w \in \mathcal{S}$. If we define

$$\bar{\mathcal{S}}_0 := \Delta \cap \mathcal{F} \cap \mathcal{S}_0,$$

then

$$\bar{\mathcal{S}}_0 \subset \mathcal{S},$$

which implies that $\bar{\mathcal{S}}_0$ is bounded by Lemma 4.2. Denote $w^{\bar{k}} := (x^{\bar{k}}, s^{\bar{k}})$. It is not difficult to see that, for each $\bar{k} \geqslant 0$, there exists a point $w^{\bar{k}^*} = (x^{\bar{k}^*}, s^{\bar{k}^*}) \in \bar{\mathcal{S}}_0$ such that

$$\|w^{\bar{k}} - w^{\bar{k}^*}\| = \min_{w \in \bar{\mathcal{S}}_0} \|w^{\bar{k}} - w\| = \operatorname{dist}(w^{\bar{k}}, \bar{\mathcal{S}}_0). \tag{24}$$

Note that when $\bar{k}$ is sufficiently large, the point $w^{\bar{k}^*}$ is also the projection of point $w^{\bar{k}}$ on $\mathcal{S}_0$. Therefore, from (21), there exists a constant $\bar{\rho} > 0$ such that

$$\operatorname{dist}(w^{\bar{k}}, \bar{\mathcal{S}}_0) = \operatorname{dist}(w^{\bar{k}}, \mathcal{S}_0) \leqslant \bar{\rho} \|H(x^{\bar{k}})\| \tag{25}$$

for any $\bar{k}$ being sufficiently large.

Corresponding to the solution point $(x^{\bar{k}^*}, s^{\bar{k}^*})$, we define four index sets:

$$\begin{aligned} A^*_{\bar{k}} &:= \left\{ i \in \mathcal{I} : x_i^{\bar{k}^*} > 0 \right\}, & C^*_{\bar{k}} &:= \left\{ i \in \mathcal{I} : x_i^{\bar{k}^*} = 0 \right\}, \\ B^*_{\bar{k}} &:= \left\{ (i,j) \in \mathcal{J} : (s^{\bar{k}^*})^i_j > 0 \right\}, & D^*_{\bar{k}} &:= \left\{ (i,j) \in \mathcal{J} : (s^{\bar{k}^*})^i_j = 0 \right\}. \end{aligned} \tag{26}$$

Obviously, $A^*_{\bar{k}}$ and $C^*_{\bar{k}}$ form a partition of the index set $\mathcal{I}$ and $B^*_{\bar{k}}$ and $D^*_{\bar{k}}$ form a partition of the index set $\mathcal{J}$.

The following lemma presents a special property of the four index sets defined above.

LEMMA 4.4. *Let N be a vertical block P_0 and R_0 matrix of type $(m_1, \ldots, m_n)$ and the index sets $A_{\bar{k}}, B_{\bar{k}}, C_{\bar{k}}, D_{\bar{k}}$ and $A^*_{\bar{k}}, B^*_{\bar{k}}, C^*_{\bar{k}}, D^*_{\bar{k}}$ be defined as in* (18) *and* (26)*, respectively. Assume that the strict complementarity condition holds. If $H(x^k) \neq 0$ for all $k \geqslant 0$, then $A_{\bar{k}} = A^*_{\bar{k}}, B_{\bar{k}} = B^*_{\bar{k}}, C_{\bar{k}} = C^*_{\bar{k}}$, and $D_{\bar{k}} = D^*_{\bar{k}}$, when $\bar{k}$ becomes sufficiently large.*

Proof.

(i) We first show that $A_{\bar{k}} \subseteq A^*_{\bar{k}}$. For any $i_0 \in A_{\bar{k}}$, i.e., $x^{\bar{k}}_{i_0} > \sqrt{\mu_{\bar{k}}}$ by using $\|H(x^{\bar{k}}, \mu_{\bar{k}})\| \leqslant \beta\mu_{\bar{k}}$ and the result (i) of Lemma 2.1, we have

$$\begin{aligned} |H_i(x^{\bar{k}})| &\leqslant |H_i(x^{\bar{k}}, \mu_{\bar{k}})| + \mu_{\bar{k}} \ln\,(m_i+1) \\ &\leqslant (\beta + \ln\,(m_i+1))\mu_{\bar{k}} \text{ for } i \in \mathcal{I}. \end{aligned} \tag{27}$$

Thus, for any $\bar{k}$ being sufficiently large,

$$\begin{aligned} \|x^{\bar{k}} - x^{\bar{k}^*}\| &\leqslant \|w^{\bar{k}} - w^{k^*}\| = \mathrm{dist}(w^{\bar{k}}, \bar{S}_0) \quad \text{(by (24))} \\ &\leqslant \bar{p}||H(x^{\bar{k}})|| \quad \text{(by (25))} \\ &= \bar{p}\sqrt{\sum_{i=1}^{n} \left|H_i(x^{\bar{k}})\right|^2} \\ &\leqslant c_5\mu_{\bar{k}}, \quad \text{(by (27))} \end{aligned} \tag{28}$$

where $c_5 := \bar{\rho}(\sum_{i=1}^{n}[\beta + \ln(m_i+1)]^2)^{1/2}$. Remembering that $x^{\bar{k}}_{i_0} > \sqrt{\mu}_{\bar{k}}$, we have

$$x^{\bar{k}^*}_{i_0} \geqslant x^{\bar{k}}_{i_0} - c_5\mu_{\bar{k}} > \sqrt{\mu_{\bar{k}}} - c_5\mu_{\bar{k}} > 0, \tag{29}$$

where the first inequality follows from (28) and the third inequality from $\lim_{\bar{k}\to\infty} \mu_{\bar{k}} = 0$. Obviously, (29) implies $i_0 \in A^*_{\bar{k}}$ and, consequently, $A_{\bar{k}} \subseteq A^*_{\bar{k}}$.

Next we show that $A^*_{\bar{k}} \subseteq A_{\bar{k}}$. For any $i_0 \in A^*_{\bar{k}}$, i.e., $x^{\bar{k}^*}_{i_0} > 0$, from the proof of Theorem 4.2, we know

$$\lim_{\bar{k}\to\infty} \mu_{\bar{k}} = 0,$$

and, by (28),

$$\|x^{\bar{k}} - x^{\bar{k}^*}\| \to 0.$$

Consequently,

$$\|x^{\bar{k}^*} - x^*\| \leqslant \|x^{\bar{k}^*} - x^{\bar{k}}\| + \|x^{\bar{k}} - x^*\| \to 0$$

as $\bar{k} \to 0$. Since $x^*_{i_0} > 0$, by the proof of Lemma 4.3, there is a constant $\xi > 0$ such that $x^{\bar{k}^*}_{i_0} \geqslant \xi > 0$, for $\bar{k}$ being sufficiently large. Note that $\mu_{\bar{k}} \to 0$ as $k \to \infty$. We know $x^{\bar{k}}_{i_0} > \sqrt{\mu_{\bar{k}}}$ for k being sufficiently large. This implies $i_0 \in A_{\bar{k}}$ and hence $A^*_{\bar{k}} \subseteq A_{\bar{k}}$.

(ii) Since $(s^{\bar{k}})^i_j = N^i_j x^{\bar{k}} + q^i_j$ and $(s^{\bar{k}^*})^i_j = N^i_j x^{\bar{k}^*} + q^i_j$, it follows that

$$\left|(s^{\bar{k}})^i_j - (s^{\bar{k}^*})^i_j\right| = \left|N^i_j x^{\bar{k}} - N^i_j x^{\bar{k}^*}\right| \leqslant ||N||\ ||x^{\bar{k}} - x^{\bar{k}^*}|| \leqslant c_s ||N|| \mu_{\bar{k}}.$$

Thus, similar to the proof of (i), we can show that $B_{\bar{k}} = B^*_{\bar{k}}$ for $\bar{k}$ being sufficiently large.

(iii) From (i), (ii) and

$$\begin{aligned} &A_{\bar{k}} \cup C_{\bar{k}} = \mathcal{I} = A^*_{\bar{k}} \cup C^*_{\bar{k}}, \quad A_{\bar{k}} \cap C_{\bar{k}} = \emptyset = A^*_{\bar{k}} \cap C^*_{\bar{k}}, \\ &B_{\bar{k}} \cup D_{\bar{k}} = \mathcal{J} = B^*_{\bar{k}} \cup C^*_{\bar{k}}, \quad B_{\bar{k}} \cap D_{\bar{k}} = \emptyset = B^*_{\bar{k}} \cap D^*_{\bar{k}}, \end{aligned}$$

it is easy to see that $C_{\bar{k}} = C^*_{\bar{k}}$ and $D_{\bar{k}} = D^*_{\bar{k}}$. □

The following result is a direct consequence of Lemma 4.4:

COROLLARY 4.1. *In the same setting of Lemma 4.4, there exists a constant* $c_6 > 0$ *such that, for any* $\bar{k}$ *being sufficiently large,*

$$x^{\bar{k}}_i \geqslant c_6 \quad \text{for all } i \in A_{\bar{k}} \text{ and } (s^{\bar{k}})^i_j \geqslant c_6 \quad \text{for all } (i.j) \in B_{\bar{k}}.$$

The following lemma characterizes a solution to VLCP in terms of the index sets. Since the proof is simple, we omit it.

LEMMA 4.5. $(x^{\bar{k}^*}, s^{\bar{k}^*}) \in \mathcal{S}$ *if and only if* $x^{\bar{k}^*} \geqslant 0$, $s^{\bar{k}^*} = Nx^{\bar{k}^*} + q \geqslant 0$ *and one of the following conditions is satisfied:*

(i) *If* $A^*_{\bar{k}} \neq \emptyset$, *then for any* $i \in A^*_{\bar{k}}$ *there exists at least one index* $(i, j) \in \mathcal{J}$ *such that* $(s^{\bar{k}^*})^i_j = 0$. *In this case,* $B^*_{\bar{k}}$ *can be either empty or non empty.*

(ii) *If* $A^*_{\bar{k}} = \emptyset$, *then* $q_{B^*_{\bar{k}}} > 0$ *and* $q_{D^*_{\bar{k}}} = 0$ *when* $B^*_{\bar{k}} \neq \emptyset$, *and* $q_{D^*_{\bar{k}}} = 0$ *when* $B^*_{\bar{k}} = \emptyset$.

From Theorem 4.2 we have known that Algorithm 3.1 generates a bounded infinite sequence of iterations, if it does not terminate in either Step 1 or Step 2. Now we are ready to show the following main result:

THEOREM 4.3. *Let* N *be a vertical block* P_0 *and* R_0 *matrix of type* $(m_l, \ldots, m_n)$. *Assume that the strict complementarity condition holds at an accumulation point of the sequence of iterations generated by Algorithm 3.1. Then Algorithm 3.1 terminates with an exact solution to VLCP in a finite number of iterations.*

Proof. Suppose that Algorithm 3.1 does not terminate in a finite number of generation, but instead generates an infinite sequence $\{(x^k, s^k, \mu_k)\}$.

Then we know $H(x^k) \neq 0$ for all $k \geqslant 0$ and the stopping criteria in Step 2 are inactive all the time. Otherwise, if $H(x^{k_0}) = 0$ for some $k_0 \geqslant 0$, then, by noting that $s^k = Nx^k + q$ for all $k \geqslant 0$, we have $(x^{k_0}, s^{k_0}) \in \mathcal{S}$ and the algorithm terminates here. Let (x^*, s^*, μ_*) be an accumulation point of the sequence of iterations $\{(x^k, s^k, \mu_k)\}$ and $\{(x_{\bar{k}}, s_{\bar{k}}, \mu_{\bar{k}})\}$ a convergent subsequence, $(x^{\bar{k}}, s^{\bar{k}}, \mu_{\bar{k}}) \to (x^*, s^*, \mu_*)$. Then (x^*, s^*) is a solution to VLCP. Suppose that (x^*, s^*) is strictly complementary. Let the index sets $A_{\bar{k}}$, $B_{\bar{k}}$, $C_{\bar{k}}$ and $D_{\bar{k}}$ be defined as in (18). Then one of the following three cases will happen:

(i) $A_{\bar{k}} \neq \emptyset$, $B_{\bar{k}} \neq \emptyset$.
(ii) $A_{\bar{k}} \neq \emptyset$, $B_{\bar{k}} = \emptyset$.
(iii) Either $A_{\bar{k}} = \emptyset$, $B_{\bar{k}} \neq \emptyset$ or $A_{\bar{k}} = \emptyset = B_{\bar{k}}$.

From Lemmas 4.4 and 4.5, if case (i) happens, then, for any $i \in A_{\bar{k}}$, we have $i \in A^*_{\bar{k}}$, i.e., $x_i^{\bar{k}*} > 0$. Since $(x^{\bar{k}*}, s^{\bar{k}*}) \in \mathcal{S}$ when $\bar{k}$ is sufficiently large, there exists an index $(i, j) \in \mathcal{J}$ such that $(s^{\bar{k}*})^i_j = 0$. Consequently, $(i, j) \in D^*_{\bar{k}}$ Therefore, $(i, j) \in D_{\bar{k}}$, i.e., $(s^{\bar{k}})^i_j \leqslant \sqrt{\mu_{\bar{k}}}$. holds for all $\bar{k}$ being sufficiently large.

Suppose that case (i) indeed happens at infinitely many $\bar{k}$. Since $A_{\bar{k}}$,$C_{\bar{k}}$ form a partition of the index set $\mathcal{I}$, and $B_{\bar{k}}$, $D_{\bar{k}}$ form a partition of the index set $\mathcal{J}$, the equation $s^{\bar{k}} = Nx^{\bar{k}} + q$ can be written as

$$\begin{pmatrix} s^{\bar{k}}_{\bar{B}_k} \\ s^{\bar{k}}_{\bar{D}_k} \end{pmatrix} = \begin{pmatrix} N_{\bar{B}_k\bar{A}_k} & N_{\bar{B}_k\bar{C}_k} \\ N_{\bar{D}_k\bar{A}_k} & N_{\bar{D}_k\bar{C}_k} \end{pmatrix} \begin{pmatrix} x^{\bar{k}}_{\bar{A}_k} \\ x^{\bar{k}}_{\bar{C}_k} \end{pmatrix} + \begin{pmatrix} q_{\bar{B}_k} \\ q_{\bar{D}_k} \end{pmatrix}. \tag{30}$$

Consider the subsequence $\{(x^{\bar{k}}, S^{\bar{k}}, \mu_{\bar{k}})\}$, (10) becomes

$$\begin{pmatrix} s^{\bar{k}}_{\bar{B}_k} + \Delta s^{\bar{k}}_{\bar{B}_k} \\ 0 \end{pmatrix} = \begin{pmatrix} N_{\bar{B}_k\bar{A}_k} & N_{\bar{B}_k\bar{C}_k} \\ N_{\bar{D}_k\bar{A}_k} & N_{\bar{D}_k\bar{C}_k} \end{pmatrix} \begin{pmatrix} x^{\bar{k}}_{\bar{A}_k} + \Delta x^{\bar{k}}_{\bar{A}_k} \\ 0 \end{pmatrix} + \begin{pmatrix} q_{\bar{B}} \\ q_{\bar{D}} \end{pmatrix}. \tag{31}$$

Subtracting (30) from (31) yields

$$\begin{pmatrix} \Delta s^{\bar{k}}_{\bar{B}_k} \\ -s^{\bar{k}}_{\bar{D}_k} \end{pmatrix} = \begin{pmatrix} N_{\bar{B}_k\bar{A}_k} & N_{\bar{B}_k\bar{C}_k} \\ N_{\bar{D}_k\bar{A}_k} & N_{\bar{D}_k\bar{C}_k} \end{pmatrix} \begin{pmatrix} \Delta x^{\bar{k}}_{\bar{A}_k} \\ -x^{\bar{k}}_{\bar{C}_k} \end{pmatrix}. \tag{32}$$

Let I and 0 denote the identity matrix and zero matrix with appropriate dimensionality, respectively. Define

$$y^{\bar{k}} := \left(\left(\Delta x^{\bar{k}}_{\bar{A}_k}\right)^T, \left(\Delta s^{\bar{k}}_{\bar{B}_k}\right)^T\right)^T, \quad Z^{\bar{k}} := \left(\left(x^{\bar{k}}_{C_k}\right)^T, \left(s^{\bar{k}}_{D_k}\right)^T\right)$$

$$P := \begin{pmatrix} -N_{\bar{B}_k\bar{A}_k} & I \\ -N_{\bar{D}_k\bar{A}_k} & 0 \end{pmatrix}, \qquad Q := \begin{pmatrix} -N_{\bar{B}_k\bar{C}_k} & 0 \\ -N_{\bar{D}_k\bar{C}_k} & I. \end{pmatrix}$$

Then (32) becomes

$$Py^{\bar{k}} = Qz^{\bar{k}}, \tag{33}$$

Therefore, when $\bar{k}$ is sufficiently large, the system (33) is solvable for y^k. By applying Gaussian elimination on (33), the linearly dependent rows and columns of P can be eliminated. Let $\bar{P}$ be a largest possible non-singular submatrix of P and $\bar{y}^{\bar{k}}$ be the corresponding variable. Then we need to solve

$$\bar{P}\bar{y}^{\bar{k}} = \bar{Q}\bar{z}^{\bar{k}},$$

where the rows of $\bar{Q}$ correspond to the rows of $\bar{P}$. Since $\bar{P}$ is non-singular, we have

$$\|\bar{y}^{\bar{k}}\| = \|\bar{P}^{-1}\bar{Q}\bar{z}^{\bar{k}}\| \leqslant \|\bar{P}^{-1}\|\|\bar{Q}\|\|\bar{z}^{\bar{k}}\|. \tag{34}$$

Noting that the definitions of $C_{\bar{k}}$, $D_{\bar{k}}$, $z^{\bar{k}}$ and the fact that $\bar{z}^{\bar{k}}$ is a subvector of $z^{\bar{k}}$, it is not difficult to see that there exists a constant $c_7 > 0$ such that $\|\bar{z}^{\bar{k}}\| \leqslant c_7\sqrt{\mu_{\bar{k}}}$. Moreover, $\|\bar{P}^{-1}\|$ is bounded above (see [29,34]). It follows from (34) that there exists a constant $c_8 > 0$ such that $\|\bar{y}^{\bar{k}}\| \leqslant c_8\sqrt{\mu_{\bar{k}}}$. Let the components of $y^{\bar{k}}$ that were removed during Gaussian elimination be zero. Then there exists a solution to (33), denoted by $\left(\left(\Delta x^{\bar{x}}_{\bar{A}_k}\right)^T, \left(\Delta s^{\bar{k}}_{\bar{B}_k}\right)^T\right)^T$, such that

$$\begin{cases} |\Delta x_i^{\bar{k}}| \leqslant c_9\sqrt{\mu_{\bar{k}}} & \forall i \in A_{\bar{k}} \\ |\Delta (s^{\bar{k}})^i_j| \leqslant c_9\sqrt{\mu_{\bar{k}}} & \forall (i,j) \in B_{\bar{k}}, \end{cases} \tag{35}$$

where $c_9 > 0$ is a constant. By using Corollary 4.1 and the definitions of $A_{\bar{k}}$, $B_{\bar{k}}$ we know that

$$\begin{cases} x_i^{\bar{k}} < c_6 & \forall i \in A_{\bar{k}} \\ (s^{\bar{k}})^i_j > c_6 & \forall (i,j) \in B_{\bar{k}}. \end{cases} \tag{36}$$

Combining (35), (36), and the fact that $\lim_{\bar{k}\to\infty} \mu_{\bar{k}} = 0$, we know

$$x^{\bar{k}}_{\bar{A}_k} + \Delta x^{\bar{k}}_{\bar{A}_k} > 0 \quad \text{and} \quad x^{\bar{k}}_{\bar{B}_k} + \Delta x^{\bar{k}}_{\bar{B}_k} > 0,$$

for all $\bar{k}$ being sufficiently large. This indicates that one of the stopping criteria in Step 2 is met for some sufficiently large $\bar{k}$. This is a contradiction. Hence $H(x^{k_0}) = 0$ for some $k_0 \geqslant 0$.

Similar arguments can be developed for cases (ii) and (iii). Hence Algorithm 3.1 terminates in a finite number of iterations.

Following Theorem 3.1, the proposed algorithm finds an exact solution to VLCP when it terminates. □

Case 2: Finite Termination under Singleton Assumption

This time let us assume that the solution set $\mathcal{S}$ of (1) is a singleton, say,

$$\mathcal{S}=\{(x^*,s^*)\}.$$

We will show that Algorithm 3.1 terminates with the unique solution even without the strict complementarity assumption.

Using (x^*,s^*), we define four index sets:

$$\begin{aligned} A^* &:= \left\{i\in\mathcal{I} : x_i^*>0\right\}, & C^* &:= \left\{i\in\mathcal{I} : x_i^*=0\right\}, \\ B^* &:= \left\{(i,j)\in\mathcal{J} : (s^*)^i_j>0\right\}, & D^* &:= \left\{(i,j)\in\mathcal{J} : (s^*)^i_j=0\right\}. \end{aligned} \tag{37}$$

Again, A^* and C^* form a partition of the index set $\mathcal{I}$, and B^* and D^* form a partition of the index set $\mathcal{J}$.

Since $\mathcal{S}=\{(x^*,s^*)\}$, the result (vi) of Lemma 2.1 assures that there is constant $c_{10}>0$ such that

$$\|x^{\bar{k}}-x^*\|\leqslant c_{10}\|H(x^{\bar{k}})\|.$$

Following a similar proof of Lemma 4.4, we can show the following result:

LEMMA 4.6. *Let N be a vertical block P_0 and R_0 matrix of type $(m_1,\dots,m_n)$ and the index sets $A_{\bar{k}}, B_{\bar{k}}, C_{\bar{k}}, D_{\bar{k}}$ and A^*,B^*,C^*,D^* be defined by* (18) *and* (37)*, respectively. Assume that the solution set of* (1) *is a singleton. If $H(x^k)\neq 0$ for all $k\geqslant 0$, then $A_{\bar{k}}=A^*$, $B_{\bar{k}}=B^*$, $C_{\bar{k}}=C^*$, and $D_{\bar{k}}=D^*$, when $\bar{k}$ becomes sufficiently large.*

Furthermore, using a similar proof of Theorem 4.3, we have the following main theorem:

THEOREM 4.4. *Let N be a vertical block P_0 and R_0 matrix of type $(m_1,\dots,m_n)$. Assume that the solution set of* (1) *is a singleton. Then Algorithm 3.1 terminates at the unique solution of VLCP in a finite number of iterations.*

5. Numerical results

To test the performance and illustrate the potential of the proposed method, we have implemented Algorithm 3.1 in MATLAB on a 1000 MHz Pentium III personal computer running Linux. The eight test problems found in Peng and Lin [30] (some of them are from the literature [8,31]) were used in our computational experiment. For easy comparison, the order of these eight problems is kept the same as in Peng and Lin's paper. For all test problems, the vertical block matrices are P_0-matrices. Moreover, Problems 1 to 5 hold the strict complementarity assumption. Problems 6, 7, and 8 were modified so that they may not be strictly complementary, but they do satisfy the singleton assumption.

The following parameters were chosen for all test problems: $\sigma_1 = 0.005$, $\sigma_2 = 0.001$, $\alpha_1 = 0.9$, $\alpha_2 = 0.85$, $\gamma = 1.0\text{e}-3$, $p = 1.0$, $\mu_0 = 0.0005$, and $\beta = \|H(x^0, \mu_0)\|/\mu_0 + 1.0\text{e}-5$. An initial point x^0 was set to be a contact vector $(a, \ldots, a)^T \in R^n$. We used the criterion $\|H(x^k)\|_1 \leqslant 1.0\text{e}-20$, where $\|\cdot\|_1$ denotes l_1-norm, to stop the algorithm.

Table 1 shows our test results of Algorithm 3.1. The second and the third columns indicate the dimensionality of the problem and the constant

Table 1. The performance of the proposed method for the test problems in [30]

Problem	n	a	$\|H(x^0)\|_1$	k^*	$\|H(x^{k^*})\|_1$	Termination
		1.0	1.0	2	0.0	Step 2 Case (i)
1	2	10.0	10.0	2	0.0	Step 2 Case (i)
		−10.0	22.0	3	0.0	Step 2 Case (i)
		1.0	9.0	2	0.0	Stopping criterion
2	6	10.0	18.0	3	0.0	Stopping criterion
		−10.0	215.0	3	0.0	Stopping criterion
	50	5.0	86.5	2	2.2e−16	Step 2 Case (i)
	100	5.0	171.5	2	1.1e−15	Step 2 Case (i)
3	200	5.0	341.5	2	3.4e−15	Step 2 Case (i)
	100	−5.0	1506	4	1.3e−15	Step 2 Case (i)
	200	−5.0	3005.9	4	2.ge−15	Step 2 Case (i)
	50	5.0	56.0	2	1.8e−12	Step 2 Case (i)
	100	5.0	106.0	2	7.8e−12	Step 2 Case (i)
4	200	5.0	206.0	2	7.3e−11	Step 2 Case (i)
	100	−5.0	2080.0	4	0.0	Stopping criterion
	200	−5.0	4180.0	4	0.0	Stopping criterion
	50	5.0	52.9	3	0.0	Stopping criterion
	100	5.0	102.95	4	0.0	Step 2 Case (i)
5	200	5.0	202.97	3	0.0	Stopping criterion
	100	−5.0	2095.0	5	0.0	Step 2 Case (i)
	200	−5.0	4195.0	5	0.0	Step 2 Case (i)
		1.0	10.0	2	0.0	Step 2 Case (i)
6	6	10.0	16.0	3	0.0	Step 2 Case (i)
		−10.0	215.0	4	0.0	Step 2 Case (i)
		1.0	10.0	2	0.0	Step 2 Case (i)
7	6	10.0	16.0	1	0.0	Stopping criterion
		−10.0	215.0	3	0.0	Step 2 Case (i)
		1.0	9.0	3	0.0	Step 2 Case (i)
8	6	10.0	18.0	3	0.0	Step 2 Case (i)
		−10.0	223.0	2	0.0	Step 2 Case (i)

used for the initial point, respectively. The l_1-norm of $H(\cdot)$ at the initial point is given in column 4. The k^* in column 5 denotes the number of iterations required to achieve the result given in column 6. The last column shows under which conditions each run is terminated.

Several observations can be made here:

1. For every test problem, the proposed method indeed finds a solution point meeting the desired accuracy in very few iterations. In particular, the exact solutions have been found in many cases, when $\|H(x^k)\|_1 = 0$. Compared with other known methods as reported in [8,30,31], our method converges in fewer iterations to achieve the known results.
2. Problems 6 to 8 were studied by Peng and Lin [30] only. Our results are always better than theirs. In particular, they were not able to solve problems 6 and 8 effectively.
3. The last column of the table shows that the algorithm terminates either by satisfying the condition at Case (i) of Step 2 or by meeting the stopping criterion. Furthermore, whenever the stopping criterion is met, the algorithm finds an exact solution. This supports the finite termination results proved in Section 4.
4. In our experiments, an overflow problem may occur in (7) when the exponential function $\exp(-x_i/\mu)$ or $\exp(-(N^i_j x + q^i_j)/\mu)$ is computed with a very large (negative) argument. This potential problem can be handled effectively by using the following equality:

$$H_i(x,\mu) = -\mu \ln\left(\exp\left(-\frac{x_i + h_i}{\mu}\right) + \sum_{j=1}^{m_i} \exp\left(-\frac{(N^i_j x + q^i_j) + h_i}{\mu}\right)\right) + h_i,$$

where $h_i \leqslant \min\left\{x_i, N^i_1 x + q^i_1, \ldots, N^i_{m_i} x + q^i_{m_i}\right\}$.

6. Conclusion

In this paper we have proposed an entropy function based Newton-type noninterior continuation method for solving vertical linear complementarity problems. We have shown that the proposed method finds an exact solution in a finite number of iterations under either the strict complementarity assumption or singleton assumption. This result is more general than those reported. The computational results that we conducted have confirmed our analysis and illustrated the potential of the proposed method.

References

1. Bertsekas, D.P. (1977), Approximation procedures based on the method of multipliers, *Journal of Optimization Theory and Applications* 23, 487–510.
2. Burke, J. and Xu, S. (1998), The global linear convergence of a non-interior path-following algorithm for linear complementarity problem, *Mathematics of Operations Reseasch* 23, 719–734.
3. Chang, P.L. (1980), A minimax approach to nonlinear programming, Ph.D. dissertation, Department of Mathematics, University of Washington, Seattle, WA.
4. Chen, B. and Chen, X. (2000), A global linear and local quadratic continuation smoothing method for variational inequalities with box constrains, *Computational Optimization and Applications* 13, 131–158.
5. Cottle, R.W. and Dantzig, G.B. (1970), A generalization of the linear complementarity problem, *Journal of Combinatorial Theory* 8, 79–90.
6. Cottle, R.W., Pang, J.S. and Stone, R.E. (1992), *The Linear Complementarity Problem*, Academic Press, Boston, MA.
7. Chen, X. and Ye, Y. (1999), On homotopy-smoothing methods for variational inequalities, *SIAM Journal on Control and Optimization* 37, 589–616.
8. Ebiefung, A.A. (1995), Nonlinear mappings associated with the generalized linear complementarity problem, *Mathematical Programming* 69, 255–268.
9. Ebiefung, A.A. and Kostreva, M.M. (1993), The generalized Leontief input–output model and its application to the choice of the new technology, *Annals of Operations Research* 44, 161–172.
10. Engelke, S. and Kanzow, C. (2000) Predictor–corrector smoothing methods for the solution of linear programming, Preprint, Department of Mathematics, University of Hamburg, Germany, March, 2000.
11. Fang, S.-C. and Tsao, H.-S.J. (1996), On the entropic perturbation and exponential penalty methods for linear programming, *Journal of Optimization Theory and Applications* 89, 461–466.
12. Fang, S.-C., Rajasekera, J.R. and Tsao, H.-S. J. (1997), *Entropy Optimization and Mathematical Programming*, Kluwer Academic Publishers, Boston/London/Dordrecht, 1997.
13. Fischer, A. and Kanzow, C. (1996), On the finite termination of an iterative method for linear complementarity problems, *Mathematical Programming* 74, 279–292.
14. Fujisawa, T. and Kuh, E.S. (1972), Piecewise-linear theory of nonlinear networks, *SIAM Journal on Applied Mathematics* 22, 307–328.
15. Goldstein, A.A. (1997), Chebyshev approximation and linear inequalities via exponentials, Report, Department of Mathematics, University of Washington, Seattle, WA.
16. Gowda, M.S. and Sznajder, R. (1994), The generalized order linear complementarity problem, *SIAM Journal of Matrix Analysis and Applications* 15, 779–795.
17. Gowda, M.S. and Sznajder, R. (1996), A generalization of the Nash equilibrium theorem on bimatrix games, *International Journal of Game Theory* 25, 1–12.
18. Hoffman, A.J. (1952), On approximate solutions of systems of linear equalities, *Journal of Research of the National Bureau of Standards* 49, 263–265.
19. Huang, Z.H. and Han, J. (2003), Non-interior continuation method for solving the monotone semidefinite complementarity problem, *Applied Mathematics and Optimization* 47, 195–211.
20. Huang, Z.H., Han, J. and Chen, Z. (2003), A predictor–corrector smoothing Newton algorithm, based on a new smoothing function, for solving the nonlinear complementarity problem with a P_0 function, *Journal of Optimization Theory and Applications* 117, 39–68.
21. Huang, Z.H., Qi, L. and Sun, D. (2004), Sub-quadratic convergence of a smoothing Newton algorithm for the P_0- and monotone LCP, *Mathematical Programming* 99, 423–441.

22. Huang, Z.H., Zhang, L. and Han, J. (2004), A hybrid smoothing–nonsmooth Newtontype algorithm yielding an exact solution of the P_0-LCP, *Journal of Computational Mathematics*, 22, 797–806.
23. Illés, T., Peng, J.M., Roos, C. and Terlaky, T. (2001), A strongly polynomial procedure yielding a maximally complementarity solution for $P_*(\kappa)$ linear complementarity problems, *SIAM Journal of Optimization* 11, 320–340.
24. Kort, B.W. and Bertsekas, D.P. (1972), A new penalty function for constrained minimization, *Proceedings of the 1972 IEEE Conference on Decision and Control*, New Orleans, Louisiana.
25. Li, X.S. (1991), An aggregate function method for nonlinear programming, *Science in China (Ser. A)* 34, 1467–1473.
26. Li, X.S. and Fang, S.-C. (1997), On the entropic regularization method for solving min–max problems with applications, *Mathematical Methods of Operations Research* 46, 119–130.
27. Mangasarian, O.L. (1979), Generalized linear complementarity problems as linear programming, *Opemtions Research Verfahren* 31, 393–402.
28. Mohan, S.R., Neogy, S.K. and Sridhar, R. (1996), The generalized linear complementarity problem revisited, *Mathematical Programming* 74, 197–218.
29. Mehrotra, S. and Ye, Y. (1993), On finding the optimal facet of linear programs, *Mathematical Programming* 62, 497–515.
30. Peng, P.J. and Lin, Z. (1999), A non-interior continuation method for generalized linear complementarity problems, *Mathematical Programming* 86, 533–563.
31. Qi, H.D. and Liao, L.Z. (1999), A smoothing Newton method for extended vertical linear complementarity problems, *SIAM Journal on Matrix Analysis and Applications* 21, 45–66.
32. Qi, H.D., Liao, L.Z. and Lin, Z. (1999), Regularized smoothing approximations to vertical nonlinear complementarity problems, *Journal of Mathematical Analysis and Applications* 230, 261–276.
33. Qi, L. and Sun, D. (2000), Improving the convergence of non-interior point algorithm for nonlinear complementarity problems, *Mathematics of Computation* 69, 283–304.
34. Schrijver, A. (1986), *Theory of Linear and Integer Programming*, Wiley, New York.
35. Sun, M. (1987), Singular control problems in bounded intervals, *Stochastics* 21, 303–344.
36. Sun, M. (1989), Monotonicity of Mangasarian's iterative algorithm for the generalized linear complementarity problem, *Journal of Mathematical Analysis and Applications* 144, 473–485.
37. Sun, D., Han, J., and Zhao, Y.B. (1998), On the finite termination of the damped-Newton algorithm for the linear complementarity problem, *Acta Mathematica Applicatae Sinica* 21, 148–154.
38. Sznajder, R. and Gowda, M.S. (1995), Generalizations of P_0 and P-properties, extended vertical and horizontal LCP's, *Linear Algebra and its Applications* 223/224, 695–716.
39. Tseng, P. (1999), Analysis of a non-interior continuation method based on Chen–Mangasarian smoothing functions for complementarity problems, In: Fukushima, M. and Qi, L. (eds.), *Reformulation-Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods*, pp. 381–404, Kluwer Academic Publishers, Boston.
40. Tseng, P. (2000), Error bounds and superlinear convergence analysis of some Newton-type methods in optimization. In: Di Pillo, G. and Giannessi, F. (eds.), *Nonlinear Optimization and Related Topics*, pp. 445–462, Kluwer Academic Publishers, Boston.
41. Tseng, P. and Bertsekas, D.P. (1993), On the convergence of the exponential multiplier method for convex programming, *Mathematical Programming* 60, 1–19.
42. Ye, Y. (1992), On the finite convergence of interior-point algorithms for linear programming, *Mathematical Programming* 57, 325–335.